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EXTERIOR DIFFERENTIAL SYSTEMS PROLONGATION AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT
Here we speak about the Applications to partial differential equations by using exterior differential system, we have shown that a technique which developed for systematically a prolongation structure a set of interrelated potential and pseudo potentials for nonlinear partial differential equations, the generalized KdV equation and Camassa-Holm are consider in this work.

KEYWORDS: Prolongation structure, Camassa-Holm, exterior differential system.

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INTRODUCTION
An exterior differential system which defines a generalized KdV equation on the transverse manifold was obtained [9]. A particular case of this equation has appeared in [10] recently. The symmetries of this equation were determined and some solutions were found as well [11]. This permitted the determination of a certain form of integrability. Also, a particular type of prolongation over a fiber bundle was found corresponding to this differential system, as well as a specific form for a Bäcklund transformation with its associated potential equation. Here, the same differential system is studied, but a fully general calculation of the prolongation over the same bundle is carried out in detail for this generalized KdV equation. This allows the prolongation structure for any case of the given parameters in the equation. For completeness, the general theory for obtaining such prolongations based on the given exterior system of differential forms that defines the equation upon sectioning to a transversal integral manifold will be outlined first. Transversal integral manifolds give solutions of the equation. Finally, this work is extended to a study of a differential system of one-forms which define an equation that includes the Camassa-Holm equation which has been of interest because it has been shown to have peaked soliton solutions. The Camassa-Holm equation has alot in common with the KdV equation, but there are significant differences as well. The KdV equation is globally well-posed when considered on a suitable Sobolevspace, while Camassa-Holm is in general not. The first derivative of a solution of the latter can become infinite in finite time. The associated prolongation equations are developed and found to be much more restrictive than the previous case. However, it is shown that at least one solution to the prolongation system can be found.

PROLONGATION
Roughly speaking, the prolongations of a differential system are the differential system obtained by adjoining to the original differential system its differential consequences. The concept of *prolongation tower*, which will be defined below, gives an abstract formulation of the operation of the prolongation. A general conjecture of ElieCartan, [2], proved by Kuranishi, [3], for a wide class of differential systems, state that an analytic differential system with independence condition it's takes a finite number of prolongations for it to be either involutive or incompatible, or has no solutions. This result is known as Cartan-Kuranishi Theorem. The proof of Cartan's conjecture has been given under a different set of hypotheses in the treatise [1]. Our purpose is to review some of the basic aspects of the

prolongation theorem. We assume that all manifolds and the differential systems under consideration are of class C^{ω}.

The prolongation tower of an exterior differential system with independence condition (I, Ω) on an n-dimensional manifold M is defined as a follows. Let $f: W_p \to M$ be an immersion and let $f_*: W_p \to G_p(M)$ denote the map into the Grassmann bundle of p-planes in TM determined by f. The Grassmann bundle $G_p(M)$ is endowed with a canonical exterior differential system $C^{(1)}$ defined the property that $f_*^{\ *}C^{(1)} = 0$ for any immersion $f: W_p \to M$. Using affine fiber coordinates $(x^i, u^\alpha, u_i^\alpha),\ 1 \le i \le p,\ 1 \le \alpha \le n$, on Grassmann bundle $G_p(M)$, the system $C^{(1)}$ is defined as the differential ideal generated by the 1-form

$$\vartheta^\alpha = du^\alpha - \sum_{i=1}^{p} u_i^\alpha dx^i. \qquad (1)$$

We choose component $V_p(I)$ of the sub-variety of $G_p(M)$ defined by the p-dimensional admissible integral elements of I and assume $V_p(I)$ to be C^{ω} manifold.

THE PROLONGATION STRUCTURES OF GENERALIZED KDV EQUATION

As the best known equation exhibiting all these phenomena, the KdV equation **(** where KdV is Korteweg-de Vries) provides an excellent prototype upon which to exercise and illustrate any new development. Accordingly, in this work we concerned with obtaining the prolongation structure of the KdV equation and illustrating its relation to the many known techniques for treating this equation. Since the analysis is performed in the perhaps unfamiliar language of Cartan's exterior differential forms [1]. Let us first give a brief introduction, defining the notation and setting up the KdV equation in terms of differential forms. While we do not emphasize the geometrical interpretation of our analysis (which is so well expressed by the differential form language), even analytically this notation is unquestionably superior for any treatment of conservation laws and integrability conditions.
These ideas are applied to a class of equation that includes the nonlinear Kortewege-de Vries equation. We write

$$v_t + (v^n)_{xxx} + \gamma \frac{n}{n+s}(v^{n+s})_x = 0. \qquad (2)$$

where, γ is a nonzero real constant. A more compact form is obtained if we set $m = n + s \neq 0$ and define a new constant $\beta = n\gamma/(n+s)$, then the (2) takes the form

$$v_t + (v^n)_{xxx} + \beta(v^m)_x = 0. \qquad (3)$$

AN EXTERIOR DIFFERENTIAL SYSTEM AND ASSOCIATED DIFFERENTIAL EQUATION

To begin the investigation, an exterior differential system which is relevant to the partial differential equation must be introduced. An exterior differential system is given which is defined over base manifold $M = \mathbb{R}^5$, which supports the differential forms. Consider the system of the 2-forms given by

$$\begin{aligned} \alpha_1 &= nu^{n-1}du \wedge dt - p\ dx \wedge dt = 0, \\ \alpha_2 &= dp \wedge dt - q\ dx \wedge dt = 0, \\ \alpha_3 &= du \wedge dt - dq \wedge dt - \gamma p u^s dx \wedge dt = 0, \end{aligned} \qquad (4)$$

then, take the differentiating forms in (4), we get

$$d\alpha_1 = -dp \wedge dx \wedge dt = dx \wedge \alpha_2,$$
$$d\alpha_2 = -dq \wedge dx \wedge dt = dx \wedge \alpha_3,$$
$$d\alpha_3 = -\gamma spu^{s-1} du \wedge dx \wedge dt - \gamma u^s dp \wedge dx \wedge dt \qquad (5)$$
$$= dx \wedge \left(\gamma \frac{s}{n} pu^{s-n} \alpha_1 + \gamma pu^s \alpha_2\right).$$

Therefore, it can be seen that of all these exterior derivatives vanish modulo $\{\alpha_j\}_{j=1}^{3}$. Any regular 2-dimensional solution manifold in the 5-dimensional space $S_2 = \{u(x,t),\ u_x = p(x,t),\ p_x(x,t) = q(x,t)\}$ satisfying a specific partial differential equation of the form (4) will annul this set of forms. The system that mentioned in (4) is integrable. The exact form of this equation which corresponds to (4) can be found explicitly by sectioning the forms into the solution manifold. It follows that

$$0 = \alpha_1 | S = ((u^n)_x - p) dx \wedge dt,$$
$$0 = \alpha_2 | S = (p_x - q) dx \wedge dt, \qquad (6)$$
$$0 = \alpha_3 | S = (u_t + q_x + \gamma pu^s) dt \wedge dx.$$

thus, the result that give us the equation (3).

DETERMINING PROLONGATION ALGEBRA

Based on the forms in system (4), the prolongation method outlined in [1] can be carried out, and the resulting system of equations can be solved quite generally. A very general prolongation corresponding to (2) can be calculated in terms of an algebra of vector fields which are defined on fibers above the base manifold that supports the forms (4). Then, to generate a prolongation algebra, the system (4) is substituted into prolongation condition $d\eta + \frac{1}{2}[\eta,\eta] = 0,\ mod\ \tilde{p}^*(I)$ which lead us to

$$A_t dt \wedge dx + A_u du \wedge dx + A_p dp \wedge dx + A_q dq \wedge dx + B_x dx \wedge dt$$
$$B_u du \wedge dt + B_p dp \wedge dt + B_q dq \wedge dt + [A,B] dx \wedge dt$$
$$= \lambda_1 (nu^{n-1} du \wedge dt - p dx \wedge dt) + \lambda_2 (dp \wedge dt - q dx \wedge dt) \qquad (7)$$
$$+ \lambda_3 (du \wedge dx - dq \wedge dt - \gamma pu^s) dx \wedge dt$$

Comparing the coefficients on the both side of two forms of (7) then, we get

$$A_u = \lambda_3, \qquad A_p = 0, \qquad A_q = 0,$$
$$B_u = n\lambda_1 u^{n-1}, \qquad B_p = \lambda_2, \qquad B_q = -\lambda_3, \qquad (8)$$
$$-A_t + B_x + [A,B] = -p\lambda_1 - q\lambda_2 - \gamma pu^s \lambda_3.$$

Subscripts indicate partial differentiation with respect to the variable indicated. Translations in x and t constitute symmetries of equation (2), and so a simplifying assumption would be to suppose that A, B are independent on(x,t). So that, $A_x = A_t = 0, B_x = B_t = 0$, means it must be that A, B are also invariant under translations in these variables. This introduces a considerable simplification into (8) reducing it to

$$A_p = 0, \qquad A_q = 0, \qquad A_u = -B_q,$$
$$-[A,B] = \frac{1}{n} u^{1-n} pB_u + qB_p - \gamma pu^s B_q. \qquad (9)$$

Example (1)

The system (9) can be reduced to a single expression which specifies the algebra of brackets of a set of basis vector field X_i. The structure of these algebra is dependent on the relative values of m and n.

Proof

The differential equations in (9) imply the following results

$$A = A(u,y), \qquad B = B(u,p,q,y), \qquad B = -qA_u(u,y) + \hat{B}(u,p,y). \quad (10)$$

Substituting B and collecting terms in q gives

$$q\left(-\frac{1}{n}u^{1-n}pA_{uu} + \hat{B}_p - [A, A_u]\right) + \frac{1}{n}pu^{1-n}\hat{B}_u + \gamma pu^s A_u + [A, \hat{B}] = 0. \quad (11)$$

Since $A, \hat{B}$ do not depend on q, then, it follows from (11)

$$\hat{B}_p = \frac{1}{n}u^{1-n}pA_{uu} + [A, A_u]. \qquad (12)$$

As A does not depend on p, this can be integrated to give $\hat{B}$

$$\hat{B}(u,p,y) = \frac{1}{2n}u^{1-n}p^2A_{uu} + [A, A_u]p + B''(u,y). \qquad (13)$$

Substituting (13) into (11) as well as $\hat{B}_u$, there results

$$\frac{1}{2n}u^{1-2n}((1-n)A_{uu} + uA_{uuu})p^3 + u^{1-n}[A, A_{uu}]p^2 + u^{1-n}B''_u p + n\gamma pu^s A_u + n\left[A, \frac{1}{2n}u^{1-n}p^2A_{uu} + [A, A_u]p + B''\right] = 0 \qquad (14)$$

Since $A,\ B''$ do not depend on p, the coefficient of p^3 must vanish giving theequation

$$(1-n)A_{uu} + uA_{uuu} = 0. \qquad (15)$$

Then, (15) can be solved for A to give

$$A(u,y) = X_1(y) + X_2(y)u + X_3(y)u^{n+1}, \qquad (16)$$

where the $X_i(y)$ are vertical vector fields. Consequently, (14) simplifies to

$$u^{1-n}\left([A, A_{uu}] + \frac{1}{2}[A, A_{uu}]\right)p^2 + \left(n\gamma u^s A_u + u^{1-n}B''_u + n[A, [A, A_u]]\right)p + n[A, B''] = 0. \qquad (17)$$

The coefficient of p^2 implies $[A, A_{uu}] = 0$, which using (16) immediately establishes two basic commutators of the vector fields X_1, X_2, and X_3

$$[X_1, X_3] = 0, \qquad [X_2, X_3] = 0 \qquad (18)$$

The coefficient of p implies the condition

$$n\gamma u^s A_u + u^{1-n}B''_u + n[A, [A, A_u]] = 0. \qquad (19)$$

Solving for B''_u and let $s = m - n$, we get

$$B''_u = n\gamma u^{m-1}A_u - nu^{n-1}[A, [A,\ A_u]] \qquad (20)$$

By differentiation (16) we get $A_u = X_2 + (n+1)u^nX_3$ and substituting to (20)

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$$B_u'' = n\gamma u^{m-1}(X_2 + (n+1)u^n X_3) - nu^{n-1}\left[X_1 + X_2 u + X_3 u^{n+1}, [X_1, X_2]\right] \quad (21)$$

Suppose at this point that $X_1,\ X_2$ do not commute with each other, then a new vector field can be defined as

$$X_7 = [X_1, X_2]. \qquad (22)$$

Sitting $X = X_3, Y = X_1$ and $Z = X_2$ in the Jacobi identity $\left[X, [Y, Z]\right] + \left[Y, [Z, X]\right] + \left[Z, [X, Y]\right] = 0$ gives

$$\left[X_3, [X_1, X_2]\right] + \left[X_1, [X_2, X_3]\right] + \left[X_2, [X_3, X_1]\right] = 0 \qquad (23)$$

Furthermore, by restitution (18) on (23) we get

$$[X_3, X_7] = 0 \qquad (24)$$

Thus, B_u'' reduces to the form

$$B_u'' = n\gamma u^{m-1}(X_2 + (n+1)u^n X_3) - nu^{n-1}(X_5 + uX_6). \qquad (25)$$

Two new commutators have been introduced to write (25) defined as

$$[X_1, X_7] = X_5, \qquad [X_2, X_7] = X_6. \qquad (26)$$

Using (26) in the Jacobi identity, the following brackets result

$$[X_2, X_5] = [X_1, X_6], \qquad [X_3, X_5] = 0. \qquad (27)$$

Finally, integrating B_u'' with respect to u yields an expression for B''

$$B'' = \frac{n}{m}\gamma u^m X_2 + \frac{n(n+1)}{n+m}\gamma u^{m+n} X_3 - u^n X_5 - \frac{n}{n+1} X_6 + X_4. \qquad (28)$$

Only one term in (17) remains to be satisfied, namely $[A, B''] = 0$. Thus substituting $A,\ B''$ into this bracket and using linearity to expand out, we have

$$\left[X_1 + uX_2 + u^{n+1}X_3, \frac{n}{m}\gamma u^m X_2 + \frac{n(n+1)}{m+n}\gamma u^{m+n} X_3 - u^n X_5 - \frac{n}{n+1}u^{n+1}X_6 + X_4\right]$$
$$= \frac{n}{m}\gamma u^m [X_1, X_2] - u^n [X_1, X_5] - \frac{n}{n+1}u^{n+1}[X_2, X_5] + [X_1, X_4] - u^{n+1}[X_2, X_5]$$
$$- \frac{n}{n+1}u^{n+2}[X_2, X_6] + u[X_2, X_4] - \frac{n}{n+1}u^{2n+2}[X_3, X_6] + u^{n+1}[X_3, X_4]$$

Therefore, the vector fields must be interrelated in such a way that the following holds among the coefficients of each power of u

$$[X_1, X_4] + \frac{n}{m}\gamma u^m [X_1, X_2] + u^{n+1}\left(-\frac{2n+1}{n+1}[X_2, X_5] + [X_3, X_4]\right)$$
$$+ u[X_2, X_4] - u^n [X_1, X_5] - \frac{n}{n+1}u^{n+2}[X_2, X_6] - \frac{n}{n+1}u^{2n+2}[X_3, X_6] = 0 \qquad (29)$$

Example (2)

There exist nontrivial algebras for the X_i specified by (18), (22), (24), (27) and the coefficients of powers of u in (29), which depend on the relative values of m and n.

Proof

It is required to equate the independent powers of u equal to zero. This has to be done on a case by case basis by putting individual restrictions on m and n, and not all cases are given.

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a) Suppose none of the powers of u in (29) are equal, hence $n \neq m \neq 1,0$. Equating each power of u to zero gives the following algebra

$$[X_3,X_6]=0, \qquad [X_2,X_4]=0, \qquad [X_1,X_4]=0.$$

At this point, X_1 and X_2 have be required to commute, since $X_7=0$ must hold. However, from (26), it follows that $X_5=X_6=0$. Moreover, $[X_1,X_3]=0$implies that X_1 and X_3 differ by a constant, hence X2 and X3 also differ by a constant. Finally, $[X_1,X_4]=0$ implies that X_1 and X_4 differ by a constant. Therefore, we can put

$$X_1=\varepsilon X, \qquad X_2=\sigma X, \qquad X_3=X, \qquad X_4=\alpha X. \qquad (30)$$

Substituting these results into A and B, they take the form

$$A=(\varepsilon+\sigma u+u^{n+1})X, \qquad (31)$$

$$B=-(\sigma+(n+1)u^n)qX+\frac{1}{2}(n+1)p^2X+\frac{n}{m}\gamma\sigma u^m X+\frac{n(n+1)}{m+n}\gamma u^{m+n}X+\alpha X$$

b) Suppose $n \neq 1$ and $m \neq 1,2,3,4$. Then the same algebra as (30) results and A, B are given by (31) with n set equal to one.

c) Suppose now that $n \neq m \neq 0,1$, then prolongation equation (29) reduce to

$$[X_1,X_4]+u[X_2,X_4]+u^{n+1}\left(-[X_2,X_5]-\frac{n}{n+1}[X_1,X_6]+[X_3,X_4]\right)$$
$$+u^n(\gamma X_7-[X_1,X_5])-\frac{n}{n+1}u^{n+2}[X_2,X_6]-\frac{n}{n+1}u^{2n+2}[X_3,X_6]=0 \qquad (32)$$

This equation is satisfied provided that the following brackets hold

$$[X_3,X_6]=0, \qquad [X_2,X_6]=0, \qquad \frac{2n+1}{n+1}[X_2,X_5]=[X_3,X_4] \qquad (33)$$
$$\gamma X_7=[X_1,X_5], \qquad [X_2,X_4]=0, \qquad [X_1,X_4]=0$$

in addition to the brackets given in (24), (26), and (27). This algebra has a simpler three elementsrealization which satisfies all the commutation relations provided that

$$X_3=0, \qquad X_4=0, \qquad X_5=\gamma X_2, \qquad X_6=X_2. \qquad (34)$$

The nonzero commutation relations are given by

$$[X_1,X_2]=X_7, \qquad [X_2,X_7]=X_2, \qquad [X_1,X_7]=-\gamma X_2 \qquad (35)$$

The algebra closes and a finite three-elements algebra results.

d) Suppose that $m=n+1\neq 0,1$, then prolongation equation (29) implies the algebra

$$[X_1,X_4]=0, \qquad \gamma\frac{n}{n+1}[X_1,X_2]-\frac{2n+1}{n+1}[X_2,X_5]+[X_3,X_4]=0, \qquad (36)$$
$$[X_1,X_5]=0, \qquad [X_2,X_4]=0, \qquad [X_2,X_6]=0, \qquad [X_3,X_6]=0$$

Recalling that (27) must be satisfied, a three element algebra results if we take

$$X_2=X_3, \qquad X_4=0, \qquad X_5=-\frac{\gamma n}{2n+1}X_1, \qquad X_6=\frac{\gamma n}{2n+1}X_2 \qquad (37)$$

There is a closed algebra in this case with three nontrivial brackets

$$[X_1,X_2]=X_7, \qquad [X_1,X_7]=-\frac{\gamma n}{2n+1}X_1, \qquad [X_2,X_7]=\frac{\gamma n}{2n+1}X_2 \qquad (38)$$

e) The linear case $m = n = 1$ generates the following bracket relation

$$[X_1,X_4] = [X_2,X_6] = [X_3,X_6] = 0,$$
$$\gamma X_7 + [X_2,X_4] - [X_1,X_5] = 0, \qquad (39)$$
$$[X_3,X_4] - [X_2,X_5] - \frac{1}{2}[X_1,X_6] = 0,$$

f) The case $m = 2, n = 1$ corresponds to the classical KdV equation and the brackets must satisfy

$$[X_2,X_6] = 0, \qquad \frac{1}{2}\gamma X_7[X_2,X_7] = \frac{n3}{2}[X_3,X_4] \qquad (40)$$
$$[X_2,X_4] - [X_1,X_5] = 0, \qquad [X_3,X_6] = 0, \qquad [X_1,X_4] = 0.$$

Since (27) must be satisfied, this system is satisfied if we put

$$X_3 = X_4 = 0, \qquad X_5 = -\frac{\gamma}{3}X_1, \qquad X_6 = \frac{\gamma}{3}X_2 (41)$$

There are three nontrivial commutators which take the form

$$[X_1,X_2] = X_7, \qquad [X_1,X_7] = -\frac{\gamma}{3}X_1, \qquad [X_2,X_7] = \frac{\gamma}{3}X_2 \qquad (42)$$

■

Now, we want toachieve a class of prolongation for the system (9), these condition imply $A = A(u,y),\ B = B(u,p,q,y)$. Let us take the following form for the vector fields A

$$A = A(u,y) = X_1 + uX_2, \qquad X_i = X_i(y), \qquad i = 1,2. \qquad (43)$$

Using $A_u = X_2$ and (4.8), A in (43) is sufficient to determine B in the form

$$B = -qX_2 + C(u,p,y). \qquad (44)$$

Thence, the second equation in (9) takes the form

$$[X_1 + uX_2, -qX_2 + C] = -\frac{p}{n}u^{1-n}C_u - qC_p - \gamma p u^s X_2.$$

Simplifying the above formula, it follows

$$\frac{p}{n}C_u + qu^{n-1}C_p = -\gamma p u^{s+n-1}X_2 + qu^{n-1}[X_1,X_2] - u^{n-1}[X_1,C] - u^n[X_2,C] \qquad (45)$$

Now, by defining the vector field $X_3 = [X_1,X_2]$, then whenever C is independent of q, we obtain form (45) that

$$C(u,p,y) = pX_3 + D(u,y). \qquad (46)$$

Substituting C in (46) into (45), we get

$$\frac{p}{n}D_u = p\{-\gamma u^{s+n-1}X_2 - u^{n-1}[X_1,X_3] - u^n[X_2,X_3]\} - u^{n-1}\{[X_1,D] - u[X_2,D]\}. \qquad (47)$$

Furthermore, the last term on (47) must vanish because D does not depend on p, then we have two condition on D

$$[X_1, D] - u[X_2, D] = 0,$$
$$\frac{1}{n} D_u = -\gamma u^{m-1} X_2 - u^{n-1}[X_1, X_3] - u^n [X_2, X_3] \qquad (48)$$

where $m = s + n$. By integrating in (48) with respect to u the second equation for D

$$D(u, y) = -\gamma \frac{n}{m} u^m X_2 - u^n [X_1, X_3] - \frac{n}{n+1} u^{n+1} [X_2, X_3] + X_4 (49)$$

Substituting D from (49) into the first equation with commutator in (48), it can simplify to the following

$$-\gamma \frac{n}{m} u^m [X_1, X_2] + u^n \big[X_1, [X_1, X_3]\big] + [X_1, X_4]$$
$$- u^{n+1} \Big\{ \frac{n}{n+1} \big[X_1, [X_2, X_3]\big] + \big[X_2, [X_1, X_3]\big] \Big\} + \frac{n}{n+1} u^{n+2} \big[X_2, [X_2, X_3]\big]$$
$$- u[X_2, X_4] = 0. \qquad (50)$$

Some of the brackets in the form (50) will vanish, if that m and n not be equal to one,

$$[X_2, X_4] = 0, \qquad [X_1, X_4] = 0$$

To satisfy these brackets, one way in which this can be done is to take $X_4 = \mu X_2$ and $X_4 = \varepsilon X_1$, from which it follows that $X_1 = \lambda X_2$, where μ, ε and λ are real constants. Moreover, substituting these results into the definition of X_3, it follows that $X_3 = 0$. Using all of these results in (50), it follows that the remaining terms in (50) vanish, hence (50) is satisfied identically and we have one solution. To summarize these results for the vector field, we have

$$X_1 = \lambda X_2, \qquad X_2 = X, \qquad X_3 = 0, \qquad X_4 = \mu X_2. \qquad (51)$$

Since there is only one independent vector field left, we have set $X = X_2$ in (4.50) in this case, the prolongation structure reduces to the following set of the vector fields

$$A = (\lambda + u) X,$$
$$B = -qX + C,$$
$$C = D = -\gamma \frac{n}{m} u^m X + \mu X = \left(-\gamma \frac{n}{m} u^m + \mu\right) X, \qquad (52)$$
$$X = X(y), \qquad \lambda, \mu \in \mathbb{R}.$$

Given the results for A, B in (52), the connection form $\widetilde{\omega}$ is given by

$$\widetilde{\omega} = dy - \left\{ (\lambda + u) dx + \left(-q - \gamma \frac{n}{m} u^m + \mu \right) dt \right\} X(y). \qquad (53)$$

The connection $\widetilde{\omega}$ can always be chosen on $\mathbb{R}$ with coordinates y and $X = \partial/\partial y$, thus

$$\widetilde{\omega} = dy - (\lambda + u) dx - \left(-q - \gamma \frac{n}{m} u^m + \mu \right) dt,$$

and the solutions of the system (4) determine transversal sections of the fiber bundle such that, substituting $q = (u^n)_{xx}$, these sections are defined by

$$y_x = \lambda + u,$$
$$y_t = -(u^n)_{xx} - \gamma \frac{n}{m} u^m + \mu \qquad (54)$$

Since (54) implies that $u = y_x - \lambda$, we can eliminate u to obtain an equation for $y = y(x,t)$

$$y_t + ((y_x - \lambda)^n)_{xx} + \gamma \frac{n}{m}(y_x - \lambda)^m - \mu = 0.$$

it follows that $\lambda = \mu = 0$, a potential equation in terms of y results

$$y_t + ((y_x)^n)_{xx} + \gamma \frac{n}{m}(y_x)^m = 0. \quad (55)$$

Although the prolongation or the solution of the vector fields (52) is not extremely complicated, in effect a Bäcklund transformation has been determined in the form of the equations presented in (54). This set of equations transforms the original equation into the form of its potential equation. Given a solution u of (2) then integrating (54) gives a corresponding solution y to (55).

PROLONGATION OF A DIFFERENTIAL SYSTEM RELATED TO THE CAMASSA-HOLM EQUATION AND THE DEGASPERIS-PROCESI EQUATIONS

It is the intention here to review some of the mathematical background which will let us study some interrelated equations which have been of interest recently. First we will give a brief introduction, defining the notation andsetting up the Camassa-Holm equation in terms of differential forms. While we do not emphasize the geometrical interpretation of our analysis (which is so well expressed by the differential form language), even analytically this notation is unquestionably superior for any treatment of conservation laws and integrability conditions.

These ideas are applied to a class of equations that includes the Camassa-Holm and Degasperis-Procesi equations. These equations are of the form:

$$(u - u_{xx})_t + u(u - u_{xx})_x + \beta(u - u_{xx})u_x = 0 \quad (56)$$

Where, $\beta = \text{constant}$, nonzero.

An exterior differential system which reproduces the given equation on the transverse manifold is developed for each case. The derivatives of the forms in this set are shown to be expressible in terms of the same forms, so the integrability of each equation is established. Finally, conservation laws for the two equations will be written down developed from the original set of one-forms.

Let us begin by introducing the system of exterior differential which is related to several equations which are of interest in mathematical physics at the moment. In particular, the Camassa-Holm and Degasperis-Procesi equations are to be included in this group. Define the following system of two forms

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge d \\ \alpha_3 &= du \wedge dx - dq \wedge dx + dq \wedge dt + (u - q) dx \wedge dt \end{aligned} \quad (57)$$

Then, by differentiating the forms in (57), we get

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = (1/q)\alpha_2 \wedge dp \\ d\alpha_2 &= -dq \wedge dx \wedge dt = \alpha_3 \wedge dx \\ d\alpha_3 &= du \wedge dx \wedge dt - dq \wedge dx \wedge dt = \alpha_3 \wedge dt \end{aligned} \quad (58)$$

Therefore, it can be seen that of all these exterior derivatives vanish modulo $\{\alpha_j\}_{j=1}^{3}$. Any regular 2-dimensional solution manifold in the 5-dimensional space $S_2 = \{u(x,t),\ u_x = p(x,t),\ p_x(x,t) = q(x,t)\}$

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satisfying a specific partial differential equation will annul this set of forms. The exact form can be found explicitly by sectioning the forms into the solution manifold. It follows that

$$0 = \alpha_1 | S = (u_x - p)dx \wedge dt$$
$$0 = \alpha_2 | S = (p_x - q)dx \wedge dt \qquad (59)$$
$$0 = \alpha_3 | S = ((u-q)_t - (u-q) - q_x)dt \wedge dx$$

thus, the result that give us the equation

$$(u - u_{xx})_t - (u - u_{xx}) - u_{xxx} = 0 \qquad (60)$$

this is the specific equation whose integrability is implied by system (57).
Consider the differential system:

$$\alpha_1 = du \wedge dt - p\, dx \wedge dt$$
$$\alpha_2 = dp \wedge dt - q\, dx \wedge dt \qquad (61)$$
$$\alpha_3 = du \wedge dx - dq \wedge dx + du \wedge dt - dq \wedge dt + (u-q)dx \wedge dt$$

Then, by differentiating the forms in (61), we get

$$d\alpha_1 = -dp \wedge dx \wedge dt = dx \wedge \alpha_2$$
$$d\alpha_2 = -dq \wedge dx \wedge dt = dx \wedge (-\alpha_3 + \alpha_1) \qquad (62)$$
$$d\alpha_3 = -dx \wedge \alpha_3$$

Upon sectioning these forms, and the equation which belong to (61) arises from the section $\alpha_3|_S = 0$ is given by

$$(u - u_{xx})_t - (u - u_{xx})_x - (u - u_{xx}) = 0 \qquad (63)$$

The final two cases which will be introduced include equations which are being actively studied at the moment

Define the following system of two forms, let β be a real, nonzero constant

$$\alpha_1 = du \wedge dt - p\, dx \wedge dt$$
$$\alpha_2 = dp \wedge dt - q\, dx \wedge dt \qquad (64)$$
$$\alpha_3 = -du \wedge dx + dq \wedge dx - \beta u\, dq \wedge dt + \beta(2u - q)du \wedge dt$$

then, by differentiating the forms in (4.69), we get

$$d\alpha_1 = dx \wedge dp \wedge dt = dx \wedge \alpha_2$$
$$d\alpha_2 = dx \wedge dq \wedge dt = (1/\beta u)dx \wedge (-\alpha_3 + \beta u \alpha_1 + \beta(u-q)\alpha_1) \qquad (65)$$
$$d\alpha_3 = 0$$

Obviously all of the (65) vanish modulo the set of the α_j in (64). Upon sectioning these forms, and the equation obtained from the restriction α_3

$$\alpha_3|_S = ((u-q)_t + \beta u u_x - \beta u q_x + \beta(u-q)u_x)dx \wedge dt$$

from sectioning α_1 and α_2, we have get

$$(u - u_{xx})_t - \beta\big(u(u - u_{xx})\big)_x = 0 \qquad (66)$$

The following system leads to an important class ofpartial differential equations which are of much current interest. The Camassa-Holm and Degasperis-Procesi equations are to be included in this group. Define the system of forms:

$$\alpha_1 = du \wedge dt - p\, dx \wedge dt$$
$$\alpha_2 = dp \wedge dt - q\, dx \wedge dt \qquad (67)$$
$$\alpha_3 = -du \wedge dx + dq \wedge dx - u\, dq \wedge dt + u du \wedge dt + \beta(u-q)du \wedge dt$$

Differentiating (67), we have:

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$$d\alpha_1 = -dp \wedge dx \wedge dt = dx \wedge \alpha_2$$
$$d\alpha_2 = dx \wedge \big(-\alpha_3 + u\big((1+\beta)u - q\big)\alpha_1\big)$$
$$d\alpha_3 = (1-\beta)dq \wedge du \wedge dt$$
$$= (1-\beta)dq \wedge \alpha_1 + (1-\beta)p\, dq \wedge dx \wedge dt \qquad (68)$$
$$= (1-\beta)dq \wedge \alpha_1 + (1-\beta)p\,(\alpha_3 + du \wedge dx) \wedge dt$$
$$= (1-\beta)dq \wedge \alpha_1 + (1-\beta)p\, dt \wedge \alpha_3 - (1-\beta)p\, dx \wedge du \wedge dt$$
$$= (1-\beta)[dq \wedge \alpha_1 + p\, dt \wedge \alpha_3 - p\, dx \wedge \alpha_1]$$

All of the details for calculating $d\alpha_3$ have been shown here. Obviouslyall of the $d\alpha_j$vanish modulo the set of α_j.

from sectioning α_1 and α_2, we have get other cases and the equation results from evaluating the section as follows:

$$0 = \alpha_1|_S = (u_x - p)dx \wedge dt$$
$$0 = \alpha_2|_S = (p_x - q)dx \wedge dt \qquad (69)$$
$$0 = \alpha_3|_S = ((u-q)_t + u(u-q)_x + \beta(u-q)u_x)dx \wedge dt$$

These results imply the partial differential equation:

$$(u-q)_t + u(u-q)_x + \beta(u-q)u_x = 0 \qquad (70)$$

then, by putting $\beta = 3\ and\ \ \rho = u - q$ the equation (70) becomes the Degasperis-Procesi equation

$$\rho_t + u\,\rho_x + 3\rho\, u_x = 0 \qquad (71)$$

again by putting $\beta = 2\ and\ \ \rho = u - q$ the equation (70) becomes the Camassa-Holm equation

$$\rho_t + u\,\rho_x + 2\rho\, u_x = 0. \qquad (72)$$

CONCLUSIONS

As a conclusion, the generalized KdV equation is considered in this paper. Using the technique prolongation structure. We observe that the corresponding to different forms of the original non linear equation. The resulting lie algebra is realized and the Backlund transformation obtained from prolongation structure is derived.

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